Integrals

Integration

Integration is the inverse process of differentiation.

If $\frac{d}{dx}$ (F(x)) = g(x), then $\int g(x)dx$ = F(x) + C

Where C is an arbitrary constant known as the constant of integration.

Geometrical interpretation of indefinite integrals:

- An indefinite integral is a collection of a family of curves, each of which is obtained by translating one of the curves parallel to itself.

Properties of Indefinite Integrals

Property – I: $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C,$

Where C is an arbitrary constant.

Property – II: If f and g are two functions, then: $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$

Property – III: If f and g are two functions, then $\int[f(x) + g(x)]dx = \int f(x)dx$ + $\int g(x)dx$

Property – IV: $\int kf(x)dx = k\int f(x)dx$ where k is any real number.

Property – V: $\int [k_1 f_1(\text{x}) + k_2 f_2(\text{x}) + k_3 f_3(\text{x}) + \ldots + k_n f_n(\text{x})]\ \text{dx} =$

$$k_1 \int f_1\, dx + k_2 \int f_2\, dx + k_3 \int f_3\, dx + \ldots. + k_n \int f_n dx$$

Comparison between Differentiation and Integration

Differentiation and integration operate on functions.

Differentiation and integration satisfy the property of linearity.

All functions are not differentiable and all functions are not integrable.

A function has unique derivative, but does not have a unique integral.

When a polynomial function is differentiated, the result is a polynomial function whose degree is one less than the degree of the original function.

When a polynomial function is integrated, the result is a polynomial function whose degree is one more than the degree of the original function.

The derivative of a function can be found at a point, whereas the integral of a function can be found over an interval.

Derivatives are useful in finding a physical quantity like the velocity of a moving particle. Integrals are useful in finding the displacement when the velocity of a particle at a time t is given.

The processes of differentiation and integration involve limits.

The processes of differentiation and integration are inverses of each other.

Integration by Substitution

Integrals of the form, $\int f[g(x)'(x)dx$

$\int f[g(x)'(x)dx$ Can be solved using the method of substitution.

Integral	Solution
$\int \tan x \, dx$	$\log\lvert\sec x\rvert + C$
$\int \cot x \, dx$	$\log\lvert\sin x\rvert + C$
$\int \sec x \, dx$	$\log\lvert\sec x + \tan x\rvert + C$
$\int cosec\ x \, dx$	$\log\lvert\operatorname{cosec} x - \cot x\rvert + C$

Integration Using Trigonometric Identities

Steps to solve integrals involving trigonometric functions:

- Use standard trigonometric identities to convert the integrated into functions whose integrals can be evaluated easily.
- Simplify the integrand.
- Perform the integration. Use method of substation, if required.

Integrals of Some Particular Functions – I

Integral	Solution
$\int \frac{dx}{x^2 - a^2}$	$\frac{1}{2a} \log \left\lvert \frac{x-a}{x+a} \right\rvert + C$
$\int \frac{dx}{a^2 - x^2}$	$\frac{1}{2a} \log \left\lvert \frac{a+x}{a-x} \right\rvert + C$
$\int \frac{dx}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a} + C$

Integrals of Some Particular Functions – II

$$\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log\left| x + \sqrt{x^2 + a^2} \right| + C$$

Integrals of Some Particular Functions – III

$$\int \frac{px+q}{ax^2+bx+c}\, dx$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx$$

Integration by Partial Function

Proper function $\frac{P(x)}{Q(x)}$

Degree P(x) < Degree Q(x)

Improper function $\frac{P(x)}{Q(x)}$

Degree P(x) > Degree Q(x)

Form of the Rational Function	Form of the Partial Function
$\frac{px+q}{(x-a)(x-b)}$, a ≠ b	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
Where x^2 + bx + c cannot be factorized further	

Integration by Parts

Method of integration by parts:

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx$$

The ILATE precedence rule is used for determining the first function.

Integrals Involving Exponential Functions

$$\int e^x [f(x) + f'(x)]dx = e^x f(x) + C$$

Integrals of Some More Types

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left|x + \sqrt{x^2 + a^2}\right| + C$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$$

Definite Integration

The area of the region bounded by the curve y = f(x), the X-axis and the coordinate x = a and x = b, where a ≤ x ≤ b is, $\int_a^b f(x)dx$.

A = $\int_a^b f(x)dx$ = (b – a)$\lim_{n\to\infty}$ × $\frac{1}{n}$ [f(a) + f(a + h) + … + f(a + (n – 1)h)]

Where h = $\frac{b-a}{n}$; as n → ∞

Fundamental Theorem of Calculus

First fundamental theorem on integral calculus:

Let the area function defined by A (x) = $\int_a^x f(x)dx$, for all x ≥ a,

Where the function assumed to continuous on

[a, b]. Then A'(x), for all x ∈ [a, b]

Second fundamental theorem on integral calculus:

Let f be a continuous function of x defined on closed interval [a, b] and g be another function such that g'(x) = f(x) for all x in the domain of f.

Then: $\int_a^b f(x)dx$ = g(b) - g(a)

Evaluation of Definite Integration by Method of the Substitution

Step to evaluate $\int_a^b f(x)dx$ by the method of substitution:

Step 1: Consider the integral without taking the given limits.

Step 2: Substitute y = f(x) or x = g(y) to reduce the given integral to a known form.

Step 3: Integrate the new integrate with respect to the new variable without placing the constant of integration.

Step 4: Write the answer in terms of the original variable by

re-substituting the new variable.

Or

Keep the integral in the new variable itself and change the l limits of the integral accordingly.

Steps to evaluate $\int_a^b f(x)dx$ by the method of substitution:

Step 5: Find the values of the answers obtained in the previous step at the given upper and lower limits.

Step 6: Subtract the value at the lower limit from the value of the upper limit to obtain the required definite integral.

Properties of Definite Integrals – I

Property -1:

$\int_a^b f(x)dx = \int_a^b f(t)dt$

Property 2:

$\int_a^b f(x)dx$ = - $\int_b^a f(x)dx$ also, $\int_b^a f(x)dx = 0$

Property 3:

If a < b, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Properties of Definite Integrals – II

Property 4: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Property 5: $\int_0^b f(x)dx = \int_0^a f(a-x)dx$

Properties of Definite Integrals – III

1. Property

- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-bx)dx$

2. Property

- $\int_0^{2a} f(x)dx$ = {$2\int_0^a f(x)dx$, if f(2a – x) = f(x) 0, if f(2a – x) = - f(x)}

3. Property

- $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if f is an even function, i.e. f(-x) = f(x)
- $\int_{-a}^a f(x)dx$ = 0 if f is an odd function, i.e. f(-x) = - f(x)